FINITE SETS ON CURVES AND SURFACES*

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ABSTRACT

A complete proof is given for Schnirelmann's theorem on the existence of a square in C^2 Jordan curves. The following theorems are then proved, using the same method: 1. On every hypersurface in R^n, C^3-diffeomorphic to S^{n-1}, there exist $2n$ points which are the vertices of a regular 2^n-cell C_n. 2. Every plane C' Jordan curve can be C' approximated by a curve on which there are $2N$ distinct points which are the vertices of a centrally symmetric $2N$-gon (angles π not excluded). 3. On every plane C^2 curve there exist 5 distinct points which are the vertices of an axially symmetric pentagon with given base angles $\alpha, \pi/2 \leqq \alpha < \pi$. (The angle at the vertex on the axis of symmetry might be π).

1. L. Schnirelmann has published [4] the following theorem: *On every simple, closed, plane Jordan curve having continuous curvature of bounded variation, one can find four points which form the vertices of a square. The same holds for any finite union of such curves.*

However, Schnirelmann's proof as printed is not quite convincing. [In the Uspehi text, the only one available to the present author, the remark on p. 38, lines 28–30, does not seem to be justified by the statement of Lemma 1. This text is posthumous and apparently not an exact reprint of the 1929 paper.]

Since this theorem (as well as the related one on systems of rhombs, §5) is of great intrinsic interest and the topological method invented for its proof seems capable of a wide range of other applications, we present here a complete proof of a slightly improved version of Schnirelmann's theorem, following as closely as possible that author's method. This is done in §§ 2 to 4. §5 deals with Schnirelmann's rhomb theorem, and §§ 6, 7 bring some new applications of Schnirelmann's method to problems in two and more dimensions.

2. The space L of oriented line elements in the plane R^2 can be identified with the cartesian product $R^2 \times S^1$. In the space of differentiable, closed, plane curves, i.e., of the differentiable maps $f: S^1 \to R^2$, we use the C' metric defined by

$$d(f_1,f_2) = \max_{t\in[0,2\pi]} \left\{ |f_1(t)-f_2(t)| + |f_1'(t)-f_2'(t)| \right\}$$

For fixed t, the expression in brackets can be used as a distance in L induced by the cartesian product. Hence it is possible to speak of a curve which is near to a line element (x, y, α) in a neighborhood of a point (x, y). We say that (x, y, α) is a line element at (x, y).

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The main tool in the proof is a perturbation lemma. For convenience, the indices i will always run from 1 to 4, and the index $4 + 1$ shall be identified to 1.

MAIN LEMMA: *At the vertices A_i of a square we choose line elements σ_i characterized by the angle μ_i between the direction of the line element and the edge $\overrightarrow{A_iA_{i+1}}$ of the square. If the rank of the matrix*

$$\Delta = \begin{bmatrix} -\cos\mu_i & \cos\mu_2 + \sin\mu_2 & -\sin\mu_3 & 0 \\ 0 & -\cos\mu_2 & \cos\mu_3 + \sin\mu_3 & -\sin\mu_4 \\ -\sin\mu_1 & 0 & -\cos\mu_3 & \cos\mu_4 + \sin\mu_4 \\ \sin\mu_1 - \cos\mu_1 & \cos\mu_2 - \sin\mu_2 & \sin\mu_3 - \cos\mu_3 & \cos\mu_4 - \sin\mu_4 \end{bmatrix}$$

is at least three, then there exist neighborhoods $V(\sigma_i) \subset L$ with the following property: On all quadrupels of analytic arcs c_i such that the tangent elements to c_i are in $V(\sigma_i)$ it is possible to find points $B_i \in c_i$ which are the vertices of a square. The square B_i is unique if $\det\Delta \neq 0$.

We select that system of coordinates for which the square A_i becomes the unit square, i.e., $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (1,1)$, $A_4 = (0,1)$. On each arc c_i we use as parameter s_i the arclength measured from some point $X_i \in c_i$. The angle of the line element of c_i at X_i and $\overrightarrow{A_iA_{i+1}}$ is denoted by μ_i'. The parametric representation of the four arcs then becomes

$$\begin{aligned} x_1 &= a_1 + \cos\mu_1' s_1 + \eta_1(s_1) \\ y_1 &= b_1 - \sin\mu_1' s_1 + \theta_1(s_1) \\ x_2 &= 1 + a_2 + \sin\mu_2' s_2 + \eta_2(s_2) \\ y_2 &= b_2 + \cos\mu_2 s_2' + \theta_2(s_2) \\ x_3 &= 1 + a_3 - \cos\mu_3 s_3' + \eta_3(s_3) \\ y_3 &= 1 + b_3 + \sin\mu_3' s_3 + \theta_3(s_3) \\ x_4 &= a_4 - \sin\mu_4 s_4' + \eta_4(s_4) \\ y_4 &= 1 + b_4 - \cos\mu_4' s_4 + \theta_4(s_4) \end{aligned}$$

The neighborhoods $V(\sigma_i)$ are characterized by a number $\varepsilon > 0$ where

$$|a_i| + |b_i| + |\mu_i - \mu_i'| < \varepsilon. \tag{1}$$

and we may add the condition $|s_i| < c\,\varepsilon$.

The η_i, θ_i are analytic functions subject to

$$\eta_i(s_i) = 0(s_i^2) \qquad \theta_i(s_i) = 0(s_i^2)$$

The points $(x_i(s_i), y_i(s_i))$ are the vertices of a square iff

$$(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2=(x_{i+2}-x_{i+1})^2+(y_{i+2}-y_{i+1})^2 \qquad (i=1,2,3)$$

$$(x_3-x_1)^2+(y_3-y_1)^2=(x_4-x_2)^2+(y_4-y_2)^2$$

or by the parametric representation,

$$(-\cos\mu_1'+\varepsilon_{11})s_1+(\cos\mu_2'+\sin\mu_2'+\varepsilon_{12})s_2+(-\sin\mu_3'+\varepsilon_{13})s_3+g_1=f_1$$

$$(2)\quad (-\cos\mu_2'+\varepsilon_{22})s_2+(\cos\mu_3'+\sin\mu_3'+\varepsilon_{23})s_3+(-\sin\mu_4'+\varepsilon_{24})s_4+g_2=f_2$$

$$(-\sin\mu_1'+\varepsilon_{31})s_1+(-\cos\mu_3'+\varepsilon_{33})s_3+(\cos\mu_4+\sin\mu_4+\varepsilon_{34})s_4+g_3=f_3$$

$$(\sin\mu_1'-\cos\mu_1'+\varepsilon_{41})s_1+(\cos\mu_2'-\sin\mu_2'+\varepsilon_{42})s_2$$
$$+(\sin\mu_3'-\cos\mu_3'+\varepsilon_{43})s_3+(\cos\mu_4'-\sin\mu_4'+\varepsilon_{44})s_4+g_4=f_4$$

where in the $f_i=f_i(a_1,\cdots,a_4,b_1,\cdots,b_4)$ we have collected all the constant terms, and in the g_i the terms of order $\geqq 2$ in the s_i. By (1), computation shows

$$(3)\qquad \varepsilon_{ij}<8\varepsilon|s_j|$$

We have to solve the system (2) for given constants a_i, b_i, μ_i. Its Jacobian J can be written

$$J=\Delta+\Delta'$$

where, by (1) and (3). the elements δ_{ij} of Δ satisfy

$$(4)\qquad \delta_{ij}<10\varepsilon|s_j|$$

CASE 1. $\det\Delta\neq 0$.

If ε is small enough, $\det J\neq 0$ for small s_i, and the system (1) has a unique solution by the inverse function theorem. (An upper bound for the s_i can be obtained by successive approximation, hence ε can be determined to justify our reasoning.)

CASE 2. $\det\Delta=0$, rank $\Delta=3$.

At least one of the three-rowed minors of Δ is $\neq 0$. Since $\det\Delta$ is linear in the trigionometric functions, we may assume without loss of generality that we have a relation

$$\det\Delta=\cos\mu_4\cdot F(\mu_1\,\mu_2,\mu_3)-\sin\mu_4\cdot F(\mu_1,\mu_2,\mu_3)=0$$

where

$$F^2+G^2\neq 0$$

Hence, for a given choice of line elements σ_1, σ_2, σ_3 there exists a unique (up to orientation) line element through A_4 which will annul $\det\Delta$.

By continuity, rank $J \geqq 3$ in some neighborhood of $\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4$ in $L \times L \times L \times L$. If $\det J \neq 0$ for the given c_i, we are back at case 1. If $\det J = 0$, we momentarily direct our attention to the case $A_4 = B_4$, and approximate c_4 by analytic arcs c_4^* through A_4 and which at that point have line elements $(0,1,\mu_4), \mu_4^* \neq \mu_4$. By the preceding paragraph,

$$\det \Delta(\mu_1, \mu_2, \mu_3, \mu_4^*) \neq 0$$

hence there exists a square inscribed in (c_1, c_2, c_3, c_4^*). All these squares have their widths bounded from above and below. By the Blaschke Auswahlsatz there exists a converging sequence of squares. The limit figure is a non-degenerate oval which must be a square inscribed in (c_1, c_2, c_3, c_4). There is no upper bound for the number of squares obtained in this way. A repetition of the process eliminates the condition $A_4 = B_4$.

3. A point $(a_0, \cdots, a_{2n}; b_0, \cdots, b_{2n})$ in $(4n + 2)$ dimensional space R^{4n+2} defines a plane curve by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + a_{n+k} \sin kt)$$

$$y(t) = \frac{b_0}{2} + \sum_{k=1}^{n} (b_k \cos kt + b_{n+k} \sin kt)$$

Let $G_0(n)$ be the open set of points which represent simple closed curves. (The image of the C' topology in the space of curves is compatible with the cartesian topology in R^{4n+2}.) Ellipses with center at the origin and axes on the coordinate axes are represented by the points.

$$f_0 : a_1 = a \quad b_{n+1} = b \quad \text{all other coordinates } 0.$$

Let $G_1(n)$ be the connected component of $G_0(n)$ which contains the (connected) set of the f_0.

By T we denote the set of points which characterize curves with an inscribed square for which rank $\Delta < 3$ (computed for the line elements of the curve at the vertices of the square.) The dimension of T is $\leqq 4n$, therefore $G_2(n) = G_1(n) - T$ is a connected set, hence is arcwise connected. $G_2(n)$ is dense in $G_1(n)$. Plane curves which can be C' approximated by curves in $G_1(n)$ (i.e., curves represented by points of $G_1(n)$) therefore also can be C' approximated by curves in $G_2(n)$.

In a fixed coordinate system in the plane, let $\theta(s)$ be the tangent angle of a C' curve as a function of its arclength. The *declension* is the difference quotient

$$d(s_1, s_2) = \frac{\theta(s_1) - \theta(s_2)}{s_1 - s_2}$$

As an average over the curvature of C^2 curves the declension is of geometric interest.[1] We are interested in the curves of *bounded declension*. However, for the Fourier approximation of these curves we have to ask that the declension $d(s_0, s)$ be of bounded variation as a function of s. If this holds for all s_0, we say that the declension is of bounded variation. A sufficient condition would be for $\theta(s)$ to satisfy a Hölder condition of exponent >2.

If the curve is closed, $x(t)$ and $y(t)$ are periodic functions. We may assume the period to be 2π. The Fourier approximation $J_n(f)$ of a periodic function $f(t)$ is

$$J_n(f) = \frac{1}{2}\int_0^{2\pi} f(z)dz + \sum_{k=1}^{n} \int_0^{2\pi} \{f(z)\cos kz \cos kt + f(z)\sin kz \sin kt\}\, dz$$

Any plane, simple, closed curve of declension of bounded variation is differentiability isotopic to an ellipse. This means that there exists a map $F : S^1 I \to R^2$ of the unit cylinder into the plane such that

a) $F(t,0) = (a\cos t,\, b\sin t)$

b) $F(t,1) = (x(t), y(t))$

c) $F(t,\alpha) = (x_\alpha(t),\ y_\alpha(t))$ is a simple, closed curve of declension of bounded variation for constant α

d) $F(t,\alpha)$ is continuous of bounded variation in the two variable t, α.

This fact is well known even in n dimensions [3]. Since $S^1 \times I$ is compact, for given $\varepsilon > 0$ there exists an n_0 such that $(J_n(x_\alpha), J_n(y_\alpha))$ is a C'-approximation of (x_α, y_α) for all α and is of uniformly bounded declension, for $n \geqq n_0$. We have shown: *Every simple, closed curve of declension of bounded variation can be C' approximated by curves in $G_1(n)$ of uniformly bounded declension, hence also by curves of $G_2(n)$ of uniformly bounded declension, for some $n \geqq n_0$ depending on curve and approximation.*

4. The theorem may now easily be proved. First we show that: *All curves given by points of $G_2(n)$ admit an inscribed square.*

$G_2(n)$ is arcwise connected. We choose an arc which connects any point $c \in G_2(n)$ to an ellipse F_0. An ellipse contains an inscribed square (of edge $2ab(a^2 + b^2)^{-1/2}$). Since nearness of points in R^{4n+2} implies nearness of the curves in C' topology, it follows from the main lemma that points near a point in $G_2(n)$ which represents a curve with an inscribed square, also represent curves with an inscribed square. The result follows from the compactness of the arc.

THEOREM: *On every simple, closed C' curve of declension of bounded variation one can find four points which form the vertices of a square.*

For any sequence $\varepsilon_j \to 0$ we can find curves c_j, represented by points $C_j \in G_2(n_i)$ which C' approximate the given curve c up to ε_j. By our last result, there exists a square S_j in c_j. The set of ovals S_j is bounded and by the Blaschke Auswahlsatz there exists a converging subsequence of the S_j. All the limit vertices must be on c. Since c is simple, the limit figure can be degenerate only if an ε-arc of $c_j (j > j_0)$ passes through the four vertices of S_j. But then the declension of c, which is approximated by the declension of the c_j, cannot be bounded at the limit point, Q.E.D.

At this point the hypothesis of bounded declension, which was a technical convenience in § 3, becomes essential. The proof still holds if the curve is C' and piecewise of bounded declension.

5. The following theorem is given by Schnirelmann as Theorem 2.

DEFINITION: We shall call *complete* a system of rhombs whose vertices are on a closed curve c, if it satisfies the following conditions:

1) Every point of c can be taken as a vertex of some rhomb of the system;

2) Any two rhombs R_0 and R_1 can be connected by a continuous one-parameter family of rhombs $R_\alpha (0 \leqq \alpha \leqq 1)$ so that a fixed vertex A_0 of the rhomb R_0 passes into a fixed vertex A_1 of the rhomb R_1;

3) None of the rhombs degenerates into a figure without interior points.

THEOREM: *There exists a complete system of rhombs in every simple, closed curve of bounded continuous curvature.*

"Bounded continuous curvature" may again be replaced by "declension bounded variation."

An analysis of the preceding proof shows the following steps:

1. The main lemma. For det $J \neq 0$ it is trivial, for det $J = 0$ depends on the fact that J is linear in the trigonometric functions of the tangent angles.

2. Therefore, the quadruples of line elements which do not admit a prolongation satisfy at least two independent conditions. This allows us to choose a connected open set $G_2(n)$ in each approximation space R^{4n+2}.

3. In the final convergence argument, the main point to be established is the non-degeneracy of the limit figure.

Rhombs are characterized by the first three conditions (2). The main lemma therefore reads: "At the vertices A_i of a rhomb we choose line elements σ_i characterized by the angle μ_i between the direction of the line element and the edge $\overrightarrow{A_i A_{i+1}}$ of the rhomb. If the rank of the matrix

$$\Delta = \begin{bmatrix} -\cos\mu_1 & \cos\mu_2 + \sin\mu_2 & -\sin\mu_3 & 0 \\ 0 & -\cos\mu_2 & \cos\mu_3 + \sin\mu_3 & -\sin\mu_4 \\ -\sin\mu_1 & 0 & -\cos\mu_3 & \cos\mu_4 + \sin\mu_4 \end{bmatrix}$$

is three, then there exist neighborhoods $V(\sigma_i) \subset L$ with the following property: On all quadrupels of analytic arcs c_i such that the tangent elements to c_i <are in $V(\sigma_i)$ it is possible to find quadrupels $B_i(t)$ which for constant to form the vertices of a rhomb. $B_i(t)$ is a continuous function of t where t is identified with

one of the arclengths s_i of the arcs c_i. For the proof one simply chooses the fixed s_i so that the resulting system has a non-vanishing Jacobian determinant.

Step two then follows as before (rank $\Delta = 2$ gives four conditions) and also step three (the theorem is trivial for ellipses) by the boundedness of the declensions. The reader may easily fill in the details.

Schnirelmann also gives a theorem about possible degeneracies if the curve is only supposed to be C'.

6. It is worthwhile to investigate n-dimensional generalizations of the square theorem. The square is the two-dimensional member both of the series of n-dimensional cubes B_n and of their duals, the n-dimensional 2^n-cells C_n. (For $n>4$, the regular simplices A_n, B_n, and C_n are the only regular polyhedra). In general, a smooth closed surface in R^3 does not contain an inscribed cube. But by Schnirelmann's method we may prove.

THEOREM: *Every C^3 hypersurface in R^n, C^3-diffeomorphic to S^{n-1}, contains $2n$ points which are the vertices of a regular C_n.*

The regular C_n has $2n$ vertices and $2n(n-1)$ edges. All its two-dimensional faces are triangles. ($n \geqq 3$). A piece of hypersurface may be described by n-1 variables; for pieces laid out about all the vertices we need $2n(n-1)$ variables. The equality of the edges is expressed by $2n(n-1)-1$ equations which can be expanded so that the Jacobian matrix contains only first powers of the trigonometric functions of the Euler angles of the elements of hypersurface. These conditions alone suffice to make the polytope a regular C_n. (This situation is parallel to that in § 5, and a deeper discussion probably would yield a proof that each closed $C^{2+\varepsilon}$ hypersurface admits a continuum of inscribed C_n, $n \geqq 3$.)

By hypothesis, the function which describes the surface, as well as its Gauss curvature, can be considered as univalent differentiable functions on the sphere S^{n-1}. Therefore they can be developed into series of spherical harmonics and, by compactness, we again have the possibility of C^2 approximation by finite polynomials of $(n-1)$ spherical harmonics. These polynomials again may be characterized by points of an open set $G_1(n)$ of a finite dimensional cartesian space, in which there is a dense, connected subset which contains ellipsoids and for which the main (prolongation) lemma holds. The theorem holds for ellipsoids. A vertex of C_k in an ellipsoid of half-axes $a_1, \cdots, a_k$ has distance ρ from the center, where (See [2])

$$\frac{1}{\rho^2} = \frac{1}{k} \sum_1^k \frac{1}{a_j^2}$$

The Gauss curvature of our surfaces is continuous, hence bounded, and uniformly bounded on the approximating surfaces to one C^3 surface. The approximation of a C^3 surface by surfaces given by spherical polynomials yields a bounded set of

$C's$ which, by the Blaschke Auswahlsatz, is relatively compact in the space of ovals in R^n. A converging sequence of $C_n's$ must converge to a non-degenerate C_n since otherwise the Gauss curvature of the approximating surfaces could not be bounded. This completes the outline of the proof. A count of constants shows that also in dimensions 3 and 4 the A_n and C_n are the only universally inscribable regular polytopes.

7. According to Schnirelmann's theorem, every smooth curve contains a square. It is well known that every oval contains a symmetric hexagon, but not every oval contains a centrally symmetric octagon. In this connection, we can prove:

THEOREM: *Every simple C' curve can be C'-approximated by a curve which has an inscribed centrally symmetric $2N$-gon.*

Let $A_i\,(i = 1, \cdots, 2N)$ be the vertices of a $2N$-gon in cyclic order. The condition of central symmetry is

a) $\overline{A_iA_{i+1}} = \overline{A_{N+i}A_{N+i+1}}$ (indices mod $2N$)

b) $\sphericalangle A_i = \sphericalangle A_{N+i}$

Because of (b), condition (b) is equivalent

(b) $\overline{A_iA_{i+2}} = \overline{A_{N+i}\,A_{N+i+2}}$

Therefore, all conditions are given by relations between distances which yield a Jacobian matrix linear in the trigonometric functions of the angles of the line elements, and a Main Lemma will hold if the number of variables s_i is not less then the number of conditions. There are $2N$ variables s_i, N conditions (a) and N conditions (b). If we start from a $2N$-gon with distinct vertices, we get $2N$-gons with distinct vertices by the limit process needed to establish the Main Lemma. However, we are not able to control the sizes of the angles, and, therefore, the theorem has to be understood in such a way that *$2N$-gons with distinct vertices but angles π are admissible.* The approximation by trigonometric polynomials and the definition of a $G_2(n)$ for whose curves an inscribed $2N$-gon exists does not present any difficulties. On the other hand, since we cannot control the angles we cannot be sure that no points will merge in the final approximation process, even for analytic curves. Therefore, the theorem cannot be improved by Schnirelmann's methods, and I would even conjecture that with every reasonable measure in the space of C' curves the curves admitting a $2N$-gon would fill a subset of measure zero.

The proof of the previous theorem shows that for the complete success of Schirelmann's method a control of at least some angles of the inscribed polygon is necessary. In this direction, we have for instance the following theorem:

THEOREM: *On every simple, closed curve of declension of bounded variation* *there are five points which are the vertices of an axially symmetric pentagon with three equal edges and fixed base angles* $\alpha \geqq \pi / 2$.

If the vertex A_1 is on the axis of symmetry, the pentagon is characterized by

$$\overline{A_1A_2} = \overline{A_1A_5}$$

$$\overline{A_2A_3} = \overline{A_3A_4} = \overline{A_4A_5}$$

$$\overline{A_2A_4} = 2\,\overline{A_2A_3}\sin\frac{\alpha}{2}$$

$$\overline{A_3A_5} = 2\,\overline{A_2A_3}\sin\frac{\alpha}{2}$$

These are five equations for the five variables s_i. The existence of the pentagon for an ellipse follows from a simple continuity argument. The reader will easily fill in the details of the proof.

NOTE. (*Added in Proof*). The full Schnirelmann Theorem, for analytic curves only, was proved in a different way by R. P. Jerrard, *Inscribed squares in plane curves*, Trans. Amer. Math. Soc. **98** 1961, 234–241 (Reference supplied by the Referee).

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